THE GAMBLER'S RUIN: THE IMPORTANCE OF A FAIR COIN

BILLY GALLAGHER AND KEVIN PENNER

Abstract. We study the Gambler's Ruin problem between two players. Without assuming that the game is played with a fair coin, we derive a formula to determine the chance of each player's ruin. If the coin is unfair, we give a method for making the game fair. We discuss the importance of having a fair coin by changing the probabilities of getting heads or tails. We find that the fairness of the coin dominates the the ratio of starting dollars between the players in determining the probability of being ruined. Even if Player A starts with less money than his opponent, if the coin is weighed in Player A's favor, then the probability that he loses the game is low.

Introduction

The Gambler's Ruin is a classic demonstration of a random walk. The game is played as follows. Two players participate. There is a total of $x + z$ dollars in the game distributed this way: Player A starts with x dollars and Player B starts with z dollars where $x \in \mathbb{N}$ and $z \in \mathbb{N}$. A coin is tossed. The probability of getting heads is $0 < p < 1$ (making the analysis nontrivial) and the probability of tails is $q = 1 - p$. If the coin shows heads, Player A wins a dollar from Player B. If the coin shows tails, Player B wins a dollar from Player A. The process of tossing a coin is continued until one player runs out of money. At this point, the game ends. The winner walks away with all the money and the loser walks away with nothing. The player that loses the game is, for obvious reasons, said to be *ruined*.

Using this game setup, in section one we derive a formula to determine the chance of each player's ruin even if the coin is unfair. In section two, if the coin happens to be unfair ($p \neq q$), we give a method for making the game fair. Section three discusses why a fair coin is important. Section four concludes the paper.

1. The Probability of Ruin

We use the game setup above. Let $P(R_x)$ be the probability that Player A is ruined (Player B takes all the money) when Player A starts with x

Date: February 25, 2009.

dollars. Also, let X_n be the amount of money Player A has after $n = 1, 2, \ldots$ coin flips. Using the rule of total probability, we get [1]:

$$P(R_x) = P(R_x|X_1 = x+1)p + P(R_x|X_1 = x-1)q. \tag{1}$$

This derivation is justified by the following. Since x and z are arbitrary, only one round of the game must occur. This is why we use X_1. Additionally, $X_1 = \{x+1, x-1\}$ since the first toss can only be heads or tails. We use the rule of total probability to study the event of being ruined, $P(R_x)$, in conjunction with some X_1 that has to be reached. After the first coin flip, Player A has $x+1$ dollars if the toss was heads (happens with probability p) and $x-1$ dollars if the toss was tails (happens with probability q). Thus $P(R_x|X_1 = x+1)$ is the probability that Player A runs out of money given that the first toss was heads, and $P(R_x|X_1 = x-1)$ is the probability that Player A runs out of money given that the first toss was tails.

To simplify our notation into terms generally used to study difference equations, let $P(R_x) = y_x$. Once again, this is the probability of eventual ruin if Player A has x dollars. We can let $P(R_x|X_1 = x+1) = y_{x+1}$ since after winning one period, Player A faces eventual ruin having $x+1$ dollars. Likewise, let $P(R_x|X_1 = x-1) = y_{x-1}$.

We can now rewrite (1) as

$$y_x = y_{x+1}p + y_{x-1}q. \tag{2}$$

Since (2) is a second order homogeneous difference equation [2], we can solve (2) using ordinary differential equation methods by using the following initial conditions:

$$P(R_0) = y_0 = 1 \text{ and } P(R_{x+z}) = y_{x+z} = 0.$$

We justify these conditions as follows. Logically, if Player A was to start with no money, he would be instantly ruined. Similarly, if Player A starts with all the money, $x+z$ dollars, he instantly wins since Player B has no money to play with.

To solve (2), we rewrite the equation to be

$$0 = y_{x+1} - \frac{1}{p}y_x + \frac{q}{p}y_{x-1}.$$

We assume there is a solution of the form $y_x = r^x$. Then we have

$$0 = r^{x+1} - \frac{1}{p}r^x + \frac{q}{p}r^{x-1}. \tag{3}$$

Dividing by r^{x-1}, we get the characteristic equation for (3):

$$0 = r^2 - \frac{1}{p}r + \frac{q}{p}.$$

Substituting $q = 1-p$, we get the roots of this equation:

$$r_1 = \tfrac{1}{2p} + \tfrac{1}{2}(\tfrac{1}{p} - 2) = \tfrac{1}{p} - 1 = \tfrac{q}{p}$$
$$r_2 = \tfrac{1}{2p} - \tfrac{1}{2}(\tfrac{1}{p} - 2) = 1.$$

Hence our solution is

$$y_x = c_1 r_1^x + c_2 r_2^x = c_1\Big(\frac{q}{p}\Big)^x + c_2. \tag{4}$$

Using the initial conditions, we know $y_0 = 1 = c_1 + c_2$ and $y_{x+z} = 0 = c_1\big(\frac{q}{p}\big)^{x+z} + c_2$. Solving this system, we get

$$c_1 = \frac{1}{1 - (\frac{q}{p})^{x+z}}$$

and

$$c_2 = \frac{(\frac{q}{p})^{x+z}}{(\frac{q}{p})^{x+z} - 1}.$$

Plugging c_1 and c_2 back into (4), we end up with an equation for the probability that Player A faces ruin:

$$y_x = P(R_x) = \frac{(\frac{q}{p})^{x+z} - (\frac{q}{p})^x}{(\frac{q}{p})^{x+z} - 1}. \tag{5}$$

However, a problem arises when $p = q = \frac{1}{2}$ (division by zero). We derive a new formula. Starting with (2), we have

$$y_x = \frac{1}{2}y_{x+1} + \frac{1}{2}y_{x-1}$$

and initial conditions $y_0 = 1$ and $y_{x+z} = 0$. To solve this difference equation, we repeat the previous procedure. Rewriting, we have

$$0 = \frac{1}{2}y_{x+1} - y_x + \frac{1}{2}y_{x-1}.$$

We assume there is a solution of the form $y_x = r^x$. Dividing by r^{x-1}, we get the characteristic equation

$$0 = (r-1)(r-1).$$

Since there is a repeated root ($r_1 = r_2 = 1$), our solution y^x is of the form

$$y_x = c_1 r_1^x + c_2 x r_2^x = c_1 + c_2 x.$$

Using the initial conditions, we easily see $c_1 = 1$ and $c_2 = \frac{-1}{x+z}$. Plugging these in, we get $P(R_x)$ when $p = q = \frac{1}{2}$:

$$y_x = P(R_x) = 1 - (\frac{x}{x+z}). \tag{6}$$

We have derived the probability of Player A's ruin. If $p \neq q$, this probability is given by (5). If $p = q = \frac{1}{2}$, this probability is given by (6). Since the players continue the game until one has lost all their money, if Player A is not ruined, then he wins all the money and Player B is ruined. Then the probability that Player A wins the game is necessarily $1 - P(R_x)$.

2. Making the Game Fair

If $p \neq q$, the coin is not fair. However, you can make a fair decision when using an unfair coin with the following method [3]: suppose you have an unfair coin where the probability of getting a head (H) is p. Toss the coin twice. If the sequence is TH, declare H. If the sequence is HT, declare T. If the sequence is HH or TT, toss the coin twice again. We propose the following:

Proposition 1. *The probability of declaring heads is $\frac{1}{2}$ using this strategy.*

Proof. We begin by finding the probability that two independent tosses are the same and the probability that two independent tosses are different. To get $P(HH \text{ or } TT)$, we have $p^2 + (1-p)^2 = 2p^2 - 2p + 1$. To get $P(HT \text{ or } TH)$, we have $2p(1-p) = 2p - 2p^2$. Now,

$$\lim_{n\to\infty} (2p^2 - 2p + 1)^n = 0$$

since

$$0 < p < 1 \Rightarrow 2p < 2 \Rightarrow 2p^2 - 2p < 0 \Rightarrow (2p^2 - 2p + 1) < 1.$$

This shows that the probability that we toss forever, never declaring heads or tails, is zero.

Now, the probability of declaring H after exactly $2n$ tosses is the same as getting TT or HH for the first $n-1$ pairs of tosses and then getting a TH. Then $P(\text{ declaring H after } 2n \text{ tosses }) = (2p^2 - 2p + 1)^{n-1} p(1-p)$. Then the probability that H is declared is given by:

$$P(H) = \sum_{n=1}^{\infty} (2p^2 - 2p + 1)^{n-1} p(1-p) = p(1-p)\left[\frac{1}{1-(2p^2-2p+1)}\right] = \frac{p(1-p)}{2p(1-p)} = \frac{1}{2}$$

By using this method, $P(H) = P(T) = \frac{1}{2}$. □

Thus, if an unfair coin is bring used, Players A and B can use this method to make the game fair. Now, Player A wins a dollar from Player B if the coin toss sequence is TH. Player A loses a dollar to Player B if the sequence is HT.

3. Results and discussion: Why use a fair coin?

We now observe some results of equations (5) and (6) in Table 1.

These results show how the probabilities and initial amounts of money affect a player's probability of ruin. These results verify intuition. If a fair coin is being used and each player starts with the same amount of money, it is equally likely that Player A will lose or win the game. When a fair coin is

p	q	x	z	$P(R_x)$
0.5	0.5	10	10	0.5
$\frac{2}{3}$	$\frac{1}{3}$	10	10	0.000975
$\frac{1}{3}$	$\frac{2}{3}$	10	10	0.999000
0.5	0.5	8	12	0.6
$\frac{2}{3}$	$\frac{1}{3}$	8	12	0.003900
$\frac{1}{3}$	$\frac{2}{3}$	8	12	0.999800
0.5	0.5	12	8	0.4
$\frac{2}{3}$	$\frac{1}{3}$	12	8	0.002430
$\frac{1}{3}$	$\frac{2}{3}$	12	8	0.996100

Table 1. Probability that Player A is ruined

used with differing initial amounts of money, each player has a proportional chance of winning the game: if player A starts with 40% of the money, he has a 40% chance of winning.

However, if Player A is given an advantage ($p > \frac{1}{2}$), it is highly unlikely that he will lose the game. Additionally, if Player A is at a disadvantage ($p < \frac{1}{2}$), it is highly likely that he will lose the game. This result holds even if Player A starts with more money than Player B, indicating the fairness of the coin has a greater impact on the probability of ruin than the ratio of starting money. Of course, this assumes that the monetary ratio is not too extreme as in the case of a casino and an individual. If a casino is fair but has vastly more than an individual, (6) indicates that ruin is still likely. In summation, making the game as fair as possible is desirable since any bias is extremely influential.

4. Conclusion

We have derived equations that determine the probability of ruin and supplied a method to make the game fair in the event of a biased coin. We showed the importance of a fair coin in participating in the Gambler's Ruin. Though varying initial amounts of money affects the game, the fairness of the coin is a much larger factor in deciding the outcome.

References

[1] W. Feller, *An Introduction to Probability Theory and Its Applications*, 3rd Ed., John Wiley and Sons, Inc., New York, 1968, p. 342-349.

[2] J. Mathews, "Module for Homogeneous Difference Equations," Retrieved Feb. 25, 2009 from http://math.fullerton.edu/mathews/c2003/ZTransformDEMod.html.

[3] M. Zhao, "STAT 1151 Midterm 1," Retrieved Fall 2008 from http://www.pitt.edu/ mez25/midterm1.pdf.