THE MONTY HALL PROBLEM AND MATHEMATICAL INTUITION

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Abstract. We present a solution to the famous Monty Hall problem. Although many people believe that a contestant will not improve his odds by switching his choice when given the chance, we provide two arguments, including one that uses probability theory, that show his chances double if he switches. The introduction of new information is the key to the problem.

1. Introduction

The Monty Hall problem is a famous probability puzzle that produces counterintuitive results. The setup is simple [4]. A contestant is presented three doors, and behind one there is a valuable prize. The contestant picks a door. The host, knowing what door the prize is behind, then always opens another door, one that the prize is not behind. This reveals to the contestant that the prize is not there. The host then gives the contestant a chance to change the door he has picked. The question is, does the contestant improve his odds of winning if he changes his choice? Surprisingly, the answer is yes. Switching doubles the probability that the contestant is right.

In Section 2, we present the assumptions of the problem and give two arguments for the solution. The first argument is intuitive. The second argument is more mathematically formal. Section 3 will discuss the result as well as the importance of information in achieving the result. Section 4 concludes the paper.

2. Analysis

2.1. **Assumptions of the Problem.** In order for the problem to be practical, the host must know which door contains the prize. This assumption is crucial to the analysis in Section 2.3. If the host randomly picks a door to reveal, the problem changes. The host might reveal the prize to the contestant, in which case the contestant will definitely switch. This is not desirable from the game show's point of view, for one-third of the time the host will give away the answer! If the host randomly picks a door the prize

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is not behind, then the contestant has a 50-50 chance of winning whether he switches or stays since the information he gets by the host opening a door is not definite but another guess. Only when the host knows which door the prize is behind does the host reveal definite information to the contestant by opening a door. From here on, the host knows what door the prize is behind. He does not reveal this door.

It is important to understand that although the contestant has two doors to choose from after the host's revelation, the doors are not randomly selected. Suppose there are three doors. If the contestant starts by choosing the door with the prize, the host can reveal either of the two remaining doors. Now suppose the contestant starts by choosing a door without the prize. The host cannot choose either of the remaining doors to reveal. He has to reveal the door that does not contain the prize. The contestant's choices after the host's revelation depend on the contestant's original choice, so the two situations *cannot* be treated independently. We cannot look only at the contestant's second choice.

2.2. **Intuitive Argument.** At the beginning of the game, the contestant has a 1 in 3 chance of selecting the right door. Say the contestant selects door A. When the host opens an incorrect door, say door B, the contestant has more information. The probability that the prize is behind one of the doors the contestant did not choose, doors B and C, is $\frac{2}{3}$. The contestant then finds out that the prize is not behind one of the two doors he did not choose, door B. These facts together imply that the probability the prize is behind door C is $\frac{2}{3}$, for the block of two doors combined still has a 2 out of 3 chance of holding the prize. The contestant should switch to door C, doubling his chances of winning since $\frac{2}{3} = 2 \cdot \frac{1}{3}$.

Another way of seeing this "block argument" is to play the game with 100 doors [1]. If the host reveals 98 incorrect doors after you have picked one, do you have a 50-50 chance with the remaining two doors? No. When you made your original choice, you had a $\frac{1}{100}$ chance of winning the prize. If your strategy is to switch doors, you switch and lose only if you chose the correct door in the beginning. This happens only 1 percent of the time. By switching you have a $1 - \frac{1}{100} = \frac{99}{100}$ chance of winning.

2.3. **Formal Argument.** The result that the contestant doubles his chances can be formalized using Bayes' Theorem [2]. Using Bayes' formula, we can calculate the probability that a certain proposition is true based on what we know. Let $P(N|I)$ denote the probability that N is true given information I. With regards to the problem, N is the set of all doors that the prize could be behind, and I is the door that the host reveals. Formally $I \in N = \{A, B, C\}$.

Let $P(A)$ be the prior probability that the prize is behind door A. Then

$$P(A) = P(B) = P(C) = \frac{1}{3}$$

since it is equally likely that the prize is behind any of the three doors. Bayes' Theorem allows us to update probabilities based on new information. Here, this information is the host revealing what door the prize is not behind.

Suppose the contestant chooses door A and the host opens door C ($I =$ door C). Hence, the prize is not behind door C, so $P(C) = 0$. To get new estimates of the prize being behind door A or B given information that the prize is not behind door C, we make use of Bayes' Theorem [3]:

$$P(A|I) = \frac{P(A) \cdot P(I|A)}{P(I)}$$

$$P(B|I) = \frac{P(B) \cdot P(I|B)}{P(I)}$$

We still have $P(A) = P(B) = \frac{1}{3}$. If the prize is behind door A, the host may reveal either B or C, so $P(I|A) = \frac{1}{2}$. In words, $P(I|A)$ is the probability the host reveals door C given that the prize is behind door A. The host could have also revealed door B, so the probability the host reveals either door is one-half. If the prize is behind door B, the host must open door C because the contestant chose A. Thus $P(I|B) = 1$. If the prize is behind door C, the host cannot open door C, so $P(I|C) = 0$. This is where the crucial assumption that the host knows what door the prize is behind applies. If the host chooses a door randomly, this probability changes.

Since A, B, C are the entire sample space, the rule of total probability tells us that

$$P(I) = P(A)P(I|A) + P(B)P(I|B) + P(C)P(I|C) = \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 = \frac{1}{2}$$

Applying Bayes' Theorem:

$$P(A|I) = \frac{P(A) \cdot P(I|A)}{P(I)} = \frac{\frac{1}{3} \cdot \frac{1}{2}}{\frac{1}{2}} = \frac{1}{3}$$

$$P(B|I) = \frac{P(B) \cdot P(I|B)}{P(I)} = \frac{\frac{1}{3} \cdot 1}{\frac{1}{2}} = \frac{2}{3}$$

The information that the host gives the contestant by revealing that the prize is not behind door C changes the probabilities of the prize being behind A or B. Remember $P(B)$ was originally $\frac{1}{3}$. After the host gives the contestant more information, the contestant's chance of winning if he sticks with A is still $\frac{1}{3}$, but the chance of winning if he switches to B is $\frac{2}{3}$. The contestant should switch.

3. Results and Discussion

We now see that since the host has knowledge about the winning door, switching doors will double the contestant's chance of victory to $\frac{2}{3}$ in the game. Information from the host affects the probabilities associated with the different doors, changing the probabilities for *prior* probabilities (make a decision without any information from the host) to *conditional* probabilities (make a decision given the host revealed a door). Although it is tempting for the contestant to concentrate only on the choice once the host opens a door, leading to the belief that there is a 50-50 chance between the two remaining doors, the contestant is not placed in a completely new situation once the host reveals a door. This ignores the introduction of new information into the game. The host gives the contestant information that changes the probabilities for each door, essentially giving the contestant two-doors-for-one if he switches. Additionally, the calculations in Section 2.3 show that the contestant's original choice changes the choice the host must make, specifically $P(I|B)$. This is crucial when applying Bayes' Theorem. The importance of the host's action and the contestant's choice cannot be understated.

4. Conclusion

In the Monty Hall problem, the chance of winning doubles when the contestant chooses to switch doors after the host has revealed a losing option. This result is not easily intuited, but the result can be formalized using the techniques above. If you are still not convinced, pull out three cards and test the strategy. The introduction of new information is key to the problem, both for the contestant and host. This information is what changes the contestant's chances, leading to the surprising result.

References

[1] K. Devlin, "Monty Hall," *The Mathematical Association of America*, Retrieved Jan. 25, 2009 from http://www.maa.org/devlin/devlin_07_03.html.

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[3] I. Miller and M. Miller, *John E. Freunds Mathematical Statistics*, 6th Ed., Prentice Hall, New Jersey, 1999, p. 63.

[4] "Monty Hall Problem," *Wikipedia*, Retrieved Jan. 25, 2009 from http://www.en.wikipedia.org/wiki/Monty_Hall_problem.